JOURNAL OF APPROXIMATION THEORY 65, 279–300 (1991)

The Distribution of Zeros of Asymptotically Extremal Polynomials

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Communicated by D. S. Lubinsky

Received January 22, 1990; revised August 23, 1990

In this paper, we study the asymptotic behavior of the zeros of a sequence of polynomials whose weighted norms have the same nth root behavior as the weighted norms for certain extremal polynomials. Our results include as special cases several of the previous results of Erdős, Freud, Jentzsch, Szegő and Blatt, Saff, and Simkani. Applications are given concerning the zeros of orthogonal polynomials over a smooth Jordan curve (in particular, on the unit circle) and the zeros of polynomials of best approximation on **R** to nonentire functions.

1. INTRODUCTION

The behavior of the zeros in the complex plane **C** of sequences of polynomials is a classical subject that has been studied by many authors. Typical examples include sequences of orthogonal polynomials [3, 26, 27], monic polynomials minimizing various norms [15, 28], and polynomials of best approximation to a fixed function [2]. In this paper, we use potential theoretic methods to prove a general theorem (Theorem 2.3) dealing with the limiting behavior of the zeros of polynomials that have asymptotically

* The research of this author was supported in part by the Center for Approximation Theory, Texas A & M University, College Station, Texas, during his visit in fall, 1988.

† The research of this author was supported by the National Science Foundation under Grant DMS-881-4026.

0021-9045/91 $3.00

minimal norms. This theorem provides a unifying method by which many of the above cited results can be obtained.

To be more precise, let w be an admissible weight on a closed set $E \subset \mathbf{C}$ (cf. Definition 2.1), let $\|\cdot\|_E$ denote the supremum norm on E, and let Π_n denote the class of all algebraic polynomials (with complex coefficients) having degree not exceeding n. For such a weight, it is known (cf. [19, 21]) that the constants

$$t_n(w, E) := \inf\{\|w^n P\|_E : P(z) = z^n + \cdots \in \Pi_n\} \tag{1.1}$$

satisfy

$$t(w, E) := \lim_{n \to \infty} [t_n(w, E)]^{1/n} = \exp(-F) \tag{1.2}$$

for a suitable constant F (see (2.9), (2.10)).

By an *asymptotically minimal norm sequence* of monic polynomials $p_n(z) = z^n + \cdots \in \Pi_n$, we mean a sequence that satisfies

$$\lim_{n \to \infty} \|w^n p_n\|_E^{1/n} = t(w, E). \tag{1.3}$$

To each $p_n = \prod_{k=1}^n (z - z_k)$, we associate the *normalized distribution measure* $\nu(p_n)$ defined by

$$\nu(p_n) := \frac{1}{n} \sum_{k=1}^{n} \delta_{z_k}, \tag{1.4}$$

where δ_{z_k} is the point distribution with total mass 1 at z_k. Roughly speaking, we shall show that a suitable balayage (sweeping) of any weak-star limit of these measures is equal to the corresponding balayage of a probability measure $\mu := \mu(w, E)$ that solves the generalized minimal energy problem

$$\min_{\sigma \in \mathcal{M}(E)} \int\int \log\{|z - t|\, w(z)\, w(t)\}^{-1}\, d\sigma(z)\, d\sigma(t),$$

where $\mathcal{M}(E)$ denotes the collection of all probability measures supported on E.

The introduction of the weight w leads to substantial generalizations of classical results because it allows the study of asymptotically minimal norm polynomials over *unbounded* sets in the plane. As applications of our main theorem we describe the behavior of zeros of orthogonal polynomials (with respect to a fixed weight) over a smooth curve in $\mathbf{C}$, and the behavior of the zeros of polynomials of best approximation to a fixed function with respect to an exponential weight on $\mathbf{R} = (-\infty, \infty)$.

In Section 2, we describe our main results. In Section 3, we discuss the above-mentioned applications. The proofs of all the new theorems in Sections 2 and 3 are given in Section 4.

2. Main Results

Our main theorem extends a result of Blatt, Saff, and Simkani [2] by incorporating a weight function and using the notion of balayage. The weight function will be assumed to be *admissible* in the sense of the following definition.

Definition 2.1. Let $E \subset \mathbf{C}$ be a closed set of positive logarithmic capacity and $w: E \to [0, \infty)$. We say that w is *admissible* if each of the following conditions holds:

(i) w is upper semi-continuous,

(ii) $E_0 := \{z \in E: w(z) > 0\}$ has positive (inner logarithmic) capacity (cf. [25]), and

(iii) if E is unbounded, then $|z|\, w(z) \to 0$ as $|z| \to \infty$, $z \in E$.

Let $\mathcal{M}(E)$ denote the class of all positive unit Borel measures whose support is contained in E. If $\sigma \in \mathcal{M}(E)$, the *weighted logarithmic energy* of σ is defined by

$$I_w(\sigma) := \int\int \log\{|z-t|\, w(z)\, w(t)\}^{-1}\, d\sigma(z)\, d\sigma(t). \tag{2.1}$$

We let $V(w, E)$ denote the minimum value of this energy, i.e.,

$$V(w, E) := \inf_{\sigma \in \mathcal{M}(E)} I_w(\sigma). \tag{2.2}$$

The *w-modified capacity* of E is then defined by (cf. [19])

$$\operatorname{cap}(w, E) := \exp(-V(w, E)). \tag{2.3}$$

If E is compact and $w \equiv 1$ on E, then $\operatorname{cap}(w, E)$ coincides with the classical logarithmic capacity of E, denoted by $\operatorname{cap}(E)$.

For an admissible weight w on a closed, but not necessarily bounded, set E, it is known (cf. [19, 21]) that there exists a unique $\mu := \mu(w, E) \in \mathcal{M}(E)$ satisfying

$$I_w(\mu) = V(w, E). \tag{2.4}$$

Moreover, $\mathscr{S} = \mathscr{S}(w, E) := \operatorname{supp}(\mu)$ is compact, $\mathscr{S} \subset \{z \in E: w(z) > 0\}$, and μ has finite logarithmic energy. We define

$$F = F(w, E) := V(w, E) - \int Q \, d\mu, \tag{2.5}$$

where

$$Q(z) := \log[1/w(z)]. \tag{2.6}$$

When E is compact, $\operatorname{cap}(E) > 0$, and $w \equiv 1$ on E, then, of course, the extremal measure μ is the *equilibrium measure* of E, which will be denoted by ν_E. In this case,

$$F(1, E) = -\log[\operatorname{cap}(E)]. \tag{2.7}$$

More generally, it is known (cf. [19, 21]) that

$$F(w, E) = -\log[\operatorname{cap}(\mathscr{S})] + \int Q \, d\nu_{\mathscr{S}}. \tag{2.8}$$

Closely related to the notion of w-modified capacity is the notion of w-modified Chebyshev constant (cf. [19, 21]). When w is an admissible weight function on a closed set $E \subset \mathbf{C}$, we define $t_n(w, E)$ as in (1.1). The *w-modified Chebyshev constant* of E is then defined by

$$t(w, E) := \lim_{n \to \infty} [t_n(w, E)]^{1/n}, \tag{2.9}$$

where the limit is known to exist (cf. [19, 21]). The connection between $t(w, E)$ and $\operatorname{cap}(w, E)$ is described by (cf. [19, 21])

$$t(w, E) = \exp(-F(w, E)) = \operatorname{cap}(w, E) \exp\left(\int Q \, d\mu(w, E)\right). \tag{2.10}$$

A theorem of Blatt, Saff, and Simkani [2], which generalizes an earlier result due to Szegő [23], asserts, in particular, that when $E \subset \mathbf{C}$ is compact, $\operatorname{cap}(E) > 0$, and E does not contain or surround a set with nonempty (2-dimensional) interior and $\{p_n(z) = z^n + \cdots\}$ is a sequence of polynomials that satisfies

$$\lim_{n \to \infty} \|p_n\|_E^{1/n} = \operatorname{cap}(E),$$

then $\nu(p_n) \to \nu_E$ in the weak-star sense, where $\nu(p_n)$ is the normalized zero measure defined in (1.4). When E encloses a set having nonempty interior, such a statement cannot be true, as the following example shows.

Let $E := \{z \in \mathbf{C} : |z| = 1\}$, $w \equiv 1$ on E. Then $t(w, E) = \operatorname{cap}(w, E) = \operatorname{cap}(E) = 1$ and ν_E is the normalized angle measure $(2\pi)^{-1}\, d\theta$. Let $p_n(z) := z^n$ and $q_n(z) := z^n - 1$. Then

$$\lim_{n \to \infty} \|p_n\|_E^{1/n} = \lim_{n \to \infty} \|q_n\|_E^{1/n} = 1.$$

We note that $\nu(q_n) \to (2\pi)^{-1}\, d\theta$ in the weak-star sense, while $\nu(p_n) \to \delta_0$, the Dirac-delta measure at zero. Nevertheless, if f is a harmonic function on $\Delta := \{z \in \mathbf{C} : |z| < 1\}$ and continuous on $\Delta \cup E$, then

$$\lim_{n \to \infty} \int f\, d\nu(p_n) = \lim_{n \to \infty} \int f\, d\nu(q_n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(e^{i\theta})\, d\theta. \tag{2.11}$$

In this paper, we shall investigate this phenomenon in greater generality, using the notion of balayage.

A property is said to hold q.e. (quasi-everywhere) if it holds everywhere except on a set of (classical) capacity zero. For a nonempty compact subset S of $\mathbf{C}$, we let $D_\infty(S)$ denote the unbounded component $\bar{\mathbf{C}} \backslash S$, $\operatorname{Pc}(S) := \bar{\mathbf{C}} \backslash D_\infty(S)$ denote its polynomial convex hull, and $\partial_\infty S$ denote its outer boundary, i.e., $\partial_\infty S := \partial D_\infty(S)$. Let $\sigma \in \mathcal{M}(\operatorname{Pc}(S))$. A measure $\hat{\sigma}$ supported on $\partial_\infty S$ is a *balayage* of σ to $\partial_\infty S$ if

$$\int \log|z - t|^{-1}\, d\sigma(t) = \int \log|z - t|^{-1}\, d\hat{\sigma}(t) \qquad \text{q.e. on } \overline{D_\infty(S)}. \tag{2.12}$$

By continuity, the equality in (2.12) holds everywhere in $D_\infty(S)$. Moreover, if $\partial_\infty S$ is regular with respect to the Dirichlet problem for $D_\infty(S)$, then σ has a balayage $\hat{\sigma}$ such that (2.12) holds at every $z \in \overline{D_\infty(S)}$.

In the sequel, we adopt the notation

$$U(\sigma, z) := \int \log|z - t|^{-1}\, d\sigma(t). \tag{2.13}$$

The following result summarizes some known properties of balayage of measures (cf. [10]):

THEOREM 2.2. *Let $S \subset \mathbf{C}$ be compact and $\sigma \in \mathcal{M}(\operatorname{Pc}(S))$. Then*

(i) *σ has a balayage to $\partial_\infty S$.*

(ii) *If $\hat{\sigma}$ and $\tilde{\sigma}$ are balayages of σ to $\partial_\infty S$, both having finite logarithmic energy, then $\hat{\sigma} = \tilde{\sigma}$.*

(iii) *If $\hat{\sigma}$ is a balayage of σ to $\partial_\infty S$ and f is harmonic in the interior of* Pc(S) *and continuous on* Pc(S), *then*

$$\int f\, d\sigma = \int f\, d\hat{\sigma}.$$

(iv) *If σ has finite logarithmic energy, then it has a balayage to $\partial_\infty S$ with finite energy.*

When σ has a balayage to $\partial_\infty S$ with finite energy, this balayage will be denoted by $[\sigma]_{\mathrm{b}}$. In our main theorem below, we shall compare $[\mu(w, E)]_{\mathrm{b}}$ with $[\nu^*]_{\mathrm{b}}$, where ν^* is any weak-star limit measure of $\{\nu(p_n)\}$. The existence of $[\nu^*]_{\mathrm{b}}$ is a part of the theorem.

In the sequel, $E \subset \mathbf{C}$ is a fixed closed set, $w: E \to [0, \infty)$ is a fixed, admissible weight function, $\mu = \mu(w, E)$, $\mathcal{S} = \mathrm{supp}(\mu)$, and $F = F(w, E)$.

THEOREM 2.3. *Let $p_n(z) = z^n + \cdots \in \Pi_n$, $n \geq 1$, be a sequence of monic polynomials.*

(a) *If, for an increasing sequence Λ of integers,*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \|w^n p_n\|_{\partial_\infty \mathcal{S}}^{1/n} \leq \exp(-F), \tag{2.14}$$

then, for every closed $A \subset D_\infty(\mathcal{S})$,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(p_n)(A) = 0. \tag{2.15}$$

Moreover, if ν^* is any weak-star limit measure of $\{\nu(p_n)\}_{n \in \Lambda}$, then $\nu^* \in \mathcal{M}(\mathrm{Pc}(\mathcal{S}))$, ν^* (as well as μ) has a balayage to $\partial_\infty \mathcal{S}$ with finite energy, and

$$[\nu^*]_{\mathrm{b}} = [\mu]_{\mathrm{b}}. \tag{2.16}$$

In particular, if f is harmonic in the interior of $\mathrm{Pc}(\mathcal{S})$ *and continuous on* $\mathrm{Pc}(\mathcal{S})$, *then*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \int f \, d\nu(p_n) = \int f \, d\mu.$$

(b) *Suppose that*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \|w^n p_n\|_{\mathcal{S}}^{1/n} = \exp(-F), \tag{2.17}$$

and also that the following interior condition holds: For any closed subset A of the interior of $\mathrm{Pc}(\mathcal{S})$,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(p_n)(A) = 0. \tag{2.18}$$

Then, in the weak-star sense,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(p_n) = \mu. \tag{2.19}$$

In particular, (2.17) *and* (2.18) *imply that* $\mathcal{S} = \partial_\infty \mathcal{S}$.

(c) *Conversely, suppose that* $\partial_\infty \mathcal{S}$ *is regular with respect to the Dirichlet problem for* $D_\infty(\mathcal{S})$, w *is continuous on* $\mathcal{S}$, *and the zeros of* $\{p_n\}_{n \in \Lambda}$ *are uniformly bounded. If every weak-star limit measure* ν^* *of* $\{\nu(p_n)\}_{n \in \Lambda}$ *has a balayage* $\hat{\nu}^*$ *to* $\partial_\infty \mathcal{S}$ *satisfying*

$$\hat{\nu}^* = [\mu]_{\mathrm{b}}, \tag{2.20}$$

then (2.14) *holds with equality.*

We note that if E is a compact set of positive capacity and w is the characteristic function on E, then $\mu = \nu_E$ is supported on $\partial_\infty E$. Hence $[\mu]_{\mathrm{b}} = \mu$ and Theorem 2.3(b) reduces to the theorem of Blatt, Saff, and Simkani [2]. Moreover, Theorem 2.3(a) is an extension of [2, Lemma 3.1]. When $E_0 \subseteq \mathbf{R}$ (where $E_0 := \{z \in E: w(z) > 0\}$), the "interior condition" of Theorem 2.3(b) is automatically satisfied and the convergence (2.19) was proved in [16] (under somewhat stronger assumptions) and yielded many known results concerning the distribution of zeros of extremal incomplete polynomials as well as of orthogonal polynomials on the whole real line.

The next result does not require that the polynomials p_n be monic or that they have precise degree n. In the statement, $\nu(p_n)$ is again defined by (1.4) (except that n is now replaced by the precise degree of p_n) and we continue with the notation used in Theorem 2.3.

THEOREM 2.4. *Let* $w: E \to [0, \infty)$ *be admissible and* $p_n \in \Pi_n$, $n \geq 1$ *be a sequence of polynomials, not necessarily monic. Let* Λ *be a subsequence of integers and assume that the following conditions hold*:

(a)

$$\limsup_{\substack{n \to \infty \\ n \in \Lambda}} \|w^n p_n\|_{\partial_\infty \mathcal{S}}^{1/n} \leq 1. \tag{2.21}$$

(b) *There is a point* $z_0 \in D_\infty(\mathcal{S})$ *such that*

$$\liminf_{\substack{n \to \infty \\ n \in \Lambda}} \left\{ \frac{1}{n} \log |p_n(z_0)| + U(\mu, z_0) - F \right\} \geq 0. \tag{2.22}$$

Let ν^* *be any weak-star limit measure of* $\{\nu(p_n)\}_{n \in \Lambda}$. *Then* $\nu^* \in \mathcal{M}(\mathrm{Pc}(\mathcal{S}))$ *and, for balayage to* $\partial_\infty \mathcal{S}$,

$$[\nu^*]_{\mathrm{b}} = [\mu]_{\mathrm{b}}. \tag{2.23}$$

Remark. Let $p_n(x) = a_n x^n + \cdots$. Then the condition (2.22) with $z_0 = \infty$ is equivalent to

$$\liminf_{\substack{n \to \infty \\ n \in \Lambda}} |a_n|^{1/n} \geq e^F. \tag{2.24}$$

Thus, if (2.22) holds for $z_0 = \infty$, then we may apply Theorem 2.3 (with p_n replaced by p_n/a_n). However, it is not difficult to construct examples where (2.22) holds at a finite point, but not at ∞. Theorem 2.4 extends a result of Grothmann [8] in the same way as Theorem 2.3 extends the result in [2].

3. Applications

Our first application of Theorem 2.3 is to describe the behavior of the zeros of certain extremal polynomials; in particular, orthogonal polynomials on the unit circle. First, we develop some notation. Let $E \subset \mathbf{C}$ be compact and σ be a positive, finite, Borel measure on E. For $0 < p \leq \infty$ and $n = 0, 1, 2, \ldots$, we define

$$e_{n,p}(E) := \min_{P \in \Pi_{n-1}} \|z^n - P(z)\|_{p,\sigma}, \tag{3.1}$$

where, for a Borel measurable function g on E,

$$\|g\|_{p,\sigma} := \begin{cases} [\int_E |g(z)|^p \, d\sigma(z)]^{1/p}, & 0 < p < \infty \\ \sup_{z \in E} |g(z)|, & p = \infty. \end{cases} \tag{3.2}$$

There exist extremal polynomials

$$T_{n,p}(z) := T_{n,p}(E, \sigma; z) = z^n + \cdots \in \Pi_n$$

satisfying

$$e_{n,p}(E) = \|T_{n,p}\|_{p,\sigma}. \tag{3.3}$$

In particular, if $p = 2$ then $T_{n,2}$ are the monic orthogonal polynomials on E with respect to $d\sigma$. When E is the unit circle $\{z \in \mathbf{C} \colon |z| = 1\}$, then we shall adopt the more conventional notation and denote $T_{n,2}$ by Φ_n. We proved in [18] the following:

Theorem 3.1. *Let*

$$\limsup_{n \to \infty} |\Phi_n(0)|^{1/n} =: \rho, \tag{3.4}$$

and Λ be any subsequence of positive integers such that

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} |\Phi_n(0)|^{1/n} = \rho. \tag{3.5}$$

If $\rho < 1$, then, in the weak-star sense,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(\Phi_n) = \nu_\rho, \tag{3.6}$$

where ν_ρ denotes the normalized angle measure on the circle $|z| = \rho$ $(\nu_0 := \delta_0)$. If

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \frac{1}{n} \sum_{k=1}^{n} |\Phi_k(0)| = 0, \tag{3.7}$$

then (3.6) *holds even in the case when $\rho = 1$.*

A version of Theorem 3.1 can be easily extended to a more general setting where E is the outer boundary of a compact set having positive capacity (e.g., a Jordan curve). The essential property of σ needed for this extension is the following

Property R. For $0 < p \leqslant \infty$,

$$\limsup_{n \to \infty} \|T_{n,p}(E, \sigma; \cdot)\|_E^{1/n} \leqslant \operatorname{cap}(E), \tag{3.8}$$

where $\|\cdot\|_E$ denotes the sup norm on E.

For example, if E is a smooth Jordan curve and, following Geronimus and Shohat [7, Chap. VI], we define the modulus of increase of σ by the formula

$$a(\sigma, \delta) := \inf \int_A d\sigma, \qquad \delta > 0, \tag{3.9}$$

where the infimum is over all Borel sets $A \subset E$ with

$$\int_A |dz| = \delta,$$

then the condition

$$\lim_{\delta \to 0^+} \delta \log a(\sigma, \delta) = 0 \tag{3.10}$$

implies Property R for all $0 < p \leqslant \infty$. More general conditions for Property R can be found in the work of Stahl and Totik [22].

THEOREM 3.2. *Let E be the outer boundary of a compact set having positive capacity and assume σ is a positive, finite, Borel measure on E for which Property R holds.*

Then, for any closed subset $A \subset D_\infty(E)$,

$$\lim_{n \to \infty} \nu(T_{n,p})(A) = 0. \tag{3.11}$$

Moreover, any limiting measure ν^* of $\{\nu(T_{n,p})\}_{n=1}^{\infty}$ has a balayage to E with finite logarithmic energy, and

$$[\nu^*]_b = \nu_E, \tag{3.12}$$

where ν_E is the equilibrium measure for E.

As a further application, we study the zeros of the polynomials of weighted best uniform approximation on the whole real line to non-entire functions. Let $\alpha \geqslant 1$, $W_\alpha(x) := \exp(-|x|^\alpha)$, $x \in \mathbf{R}$, and $C_0(\mathbf{R})$ denote the class of all continuous functions on $\mathbf{R}$ vanishing at infinity. For $W_\alpha f \in C_0(\mathbf{R})$ we define

$$\varepsilon_n(\alpha, f) := \min_{P \in \Pi_n} \|W_\alpha(f - P)\|_{\mathbf{R}}, \qquad n = 0, 1, \ldots . \tag{3.13}$$

It is easy to see that for each n, there exists a unique $p_n^* := p_n^*(\alpha, f) \in \Pi_n$ such that

$$\|W_\alpha(f - p_n^*)\|_{\mathbf{R}} = \varepsilon_n(\alpha, f). \tag{3.14}$$

Moreover, since $\alpha \geqslant 1$, $\varepsilon_n(\alpha, f) \to 0$ as $n \to \infty$ (cf. [24]). We define

$$q_n^*(z) := p_n^*((n/\lambda_\alpha)^{1/\alpha} z), \qquad z \in \mathbf{C}, \tag{3.15}$$

where

$$\lambda_\alpha := \frac{\Gamma(\alpha)}{2^{\alpha-2}\{\Gamma(\alpha/2)\}^2}. \tag{3.16}$$

It is known [15, 16] that, with $w_\alpha(x) := \exp(-|x|^\alpha/\lambda_\alpha)$, the support of the equilibrium measure $\mu(w_\alpha, \mathbf{R})$ is the interval $[-1, 1]$. Moreover, $\mu(w_\alpha, \mathbf{R})$ is given by

$$\mu(w_\alpha, \mathbf{R})(B) := \frac{\alpha}{\pi} \int_B \int_{|t|}^1 \frac{y^{\alpha-1}}{\sqrt{y^2 - t^2}}\, dy\, dt, \tag{3.17}$$

where B is any Borel subset of $[-1, 1]$.

THEOREM 3.3. *Let $\alpha > 1$, $W_\alpha f \in C_0(\mathbf{R})$, and suppose*

$$\limsup_{n \to \infty} \{\varepsilon_n(\alpha, f)\}^{1/n} = 1. \tag{3.18}$$

If q_n^* are the polynomials defined by (3.15), *then $\nu(q_n^*)$ converges weak-star to $\mu(w_\alpha, \mathbf{R})$ as $n \to \infty$ through a subsequence $\Lambda = \Lambda(\alpha, f)$ of positive integers. In particular, if $\alpha > 1$ and f does not admit an extension to the complex plane as entire function, then* (3.18) *is satisfied.*

We observe that, in contrast with a theorem of Blatt, Saff, and Simkani [2], we do not require the function to have a singularity on the interval where it is approximated. In fact, (3.18) may be satisfied even when f is an entire function. In view of the results in [14], such a function must necessarily be of order at least α, or when its order is α, the type must be at least 1. The precise characterization of functions satisfying (3.18) is not yet known.

4. PROOFS

We begin by recalling some definitions and developing some notations. Let $\bar{\mathbf{C}} := \mathbf{C} \cup \{\infty\}$, G be an open connected subset of $\bar{\mathbf{C}}$, $\infty \in G$, ∂G be a compact subset of $\mathbf{C}$, and $\operatorname{cap}(\partial G) > 0$. The *Green's function* with pole at ∞ for G is defined by the formula

$$g(z; \infty, G) := \int \log|z - t| \, d\nu_{\partial G}(t) - \log[\operatorname{cap}(\partial G)]. \tag{4.1}$$

If $z_0 \in G$, $z_0 \neq \infty$, then the Green's function for G with pole at z_0 is defined by

$$g(z; z_0, G) := g\left(\frac{1}{z - z_0}; \infty, D\right), \tag{4.2}$$

where D is the image of G under the mapping $z \to (z - z_0)^{-1}$. We note that $g(z; z_0, G)$ is continuous (in the wide sense), positive in G, harmonic in $G \setminus \{z_0\}$, and satisfies

$$\lim_{\substack{z \to \zeta \\ z \in G}} g(z; z_0, G) = 0 \qquad \text{for q.e.} \quad \zeta \in \partial G. \tag{4.3}$$

For each $\tau > 1$, set

$$G_\tau := \{z \in G \colon g(z; \infty, G) > \log \tau\}. \tag{4.4}$$

The following lemma gives an estimate on the number of zeros of a polynomial in $(D_\infty(S))_\tau$, where we continue with the notation developed in Section 2.

LEMMA 4.1. *Let* w *be an admissible weight function,* $p_n(z) = z^n + \cdots \in \Pi_n$, *where* $n \geq 1$ *is a positive integer. Let* $G := D_\infty(\mathcal{S})$, $\tau > 1$, *and* $v_n := v(p_n)$ *be the zero measure associated with* p_n. *Then*

$$v_n(G_\tau) \leq \frac{1}{\log \tau} \left\{ \frac{1}{n} \log \| p_n w^n \|_{\partial_\infty \mathcal{S}} + F \right\}. \tag{4.5}$$

When $E \subset \mathbf{R}$ and p_n is a suitable extremal polynomial, it is shown in [13] that the expression in parentheses in (4.5) can be estimated by $c(\log n)/n$ in many interesting cases. When w is the characteristic function of $[-1, 1]$, the same quantity appears as an estimate in the discrepancy theorems of Erdős and Turán and Ganelius (cf. [4, 6] and also [1]).

The proof of Lemma 4.1 is essentially the same as the proof of inequality (3.7) in [2] except that, instead of $g_{K^*}(z, \infty)$, we use $\int \log|z-t| \, d[\mu]_b(t)$, where μ is the equilibrium measure as in Theorem 2.3. The following proposition summarizes certain technical details needed in the proof of Lemma 4.1.

PROPOSITION 4.2. *For any positive integer n, if* $P \in \Pi_n$ *and*

$$|w(z)^n P(z)| \leq M \qquad \text{q.e. on } \partial_\infty \mathcal{S}, \tag{4.6}$$

then

$$|P(z)| \leq M \exp\left[n \int \log|z-t| \, d[\mu]_b(t) + nF \right], \qquad z \in \mathbf{C}, \tag{4.7}$$

where $[\mu]_b$ *is the balayage of finite energy of* μ *to* $\partial_\infty \mathcal{S}$.

Proof. It is known (cf. [19, 21]) that

$$\int \log|z-t| \, d\mu(t) = Q(z) - F \qquad \text{q.e. on } \mathcal{S}, \tag{4.8}$$

where $Q = \log(1/w)$. Moreover, q.e. on $\partial_\infty \mathcal{S}$,

$$\int \log|z-t| \, d[\mu]_b(t) = \int \log|z-t| \, d\mu(t) \tag{4.9}$$

[10, Chap. IV]. In view of (4.8) and (4.9), the inequality (4.6) can be rewritten as

$$\frac{1}{n}\log|P(z)| - \int \log|z-t|\, d[\mu]_b(t) \leqslant \frac{1}{n}\log M + F \qquad \text{q.e. on } \partial_\infty \mathcal{S}. \tag{4.10}$$

Inequality (4.7) now follows from the maximum principle for potentials (cf. [21]). ∎

Proof of Lemma 4.1. Set

$$t(z) := \log|p_n(z)| + nU([\mu]_b, z) + \sum_{k\in I} g(z; z_k, G), \tag{4.11}$$

where $\{z_k\}_{k=1}^n$ are the zeros of p_n, $I := \{k : 1 \leqslant k \leqslant n, z_k \in G_\tau\}$ and $U([\mu]_b, z)$ is defined as in (2.13). Since $g(z; z_k, G) + \log|z - z_k|$ is harmonic in G, including at z_k, the function t is subharmonic in G (including at ∞). Moreover, in view of Proposition 4.2,

$$\limsup_{\substack{z\to\zeta \\ z\in G}} t(z) \leqslant \log \|p_n w^n\|_{\partial_\infty \mathcal{S}} + nF \tag{4.12}$$

for quasi-all $\zeta \in \partial_\infty \mathcal{S} = \partial G$. So, by the maximum principle,

$$t(\infty) \leqslant \log \|p_n w^n\|_{\partial_\infty \mathcal{S}} + nF. \tag{4.13}$$

But, if $k \in I$, then $z_k \in G_\tau$ and

$$g(\infty; z_k, G) = g(z_k; \infty, G) > \log \tau.$$

Since p_n is monic, this gives

$$t(\infty) \geqslant n\nu_n(G_\tau) \cdot \log \tau. \tag{4.14}$$

The estimate (4.5) now follows from (4.13) and (4.14). ∎

In order to prove Theorem 2.3, we need another fact about $U(\mu, z)$.

PROPOSITION 4.3 [21]. *If $z_0 \in \mathcal{S}$ is any point where*

$$U(\mu, z_0) + Q(z_0) = F, \tag{4.15}$$

then $U(\mu, z)$ is continuous at z_0. Consequently, $U(\mu, z)$ is continuous q.e. on **C**. *Furthermore, if w is continuous,* (4.15) *holds at every regular point of $\partial_\infty \mathcal{S}$.*

Proof of Theorem 2.3(a). The conclusion (2.15) follows from Lemma 4.1 and (2.14). To prove the second assertion, let ν^* be any weak-star limit measure of $\{\nu(p_n)\}_{n\in\Lambda}$, say

$$\lim_{\substack{n\to\infty\\ n\in\Lambda_1}} \nu(p_n)=\nu^*, \qquad \Lambda_1\subset\Lambda.$$

By (2.15), we have $\operatorname{supp}(\nu^*)\subset \operatorname{Pc}(\mathscr{S})$.

Let

$$\mathscr{S}_1 := \{z\in \overline{D_\infty(\mathscr{S})}: \operatorname{dist}(z, \partial_\infty\mathscr{S})\leqslant 1\}.$$

We fix an arbitrary point $z_0\in\partial_\infty\mathscr{S}$ and construct a polynomial $\hat{p}_n$ from p_n as follows. Let $\{z_{i,n}\}_1^n$ be the zeros of p_n and let k_n denote the number of these zeros that lie in $D_\infty(\mathscr{S}_1)$. Then set

$$\hat{p}_n(z) := (z-z_0)^{k_n}\Pi_{z_{i,n}\notin D_\infty(\mathscr{S}_1)}(z-z_{i,n}).$$

Since $k_n/n\to 0$ as $n\to\infty$, it is easy to see that ν^* is also a weak-star limit of $\nu(\hat{p}_n)$ and that $\{\hat{p}_n\}$ also satisfies (2.14).

From (2.14) and Proposition 4.2, it follows that

$$U([\mu]_b, z)\leqslant \varepsilon_n + U(\nu(\hat{p}_n), z), \qquad z\in\mathbf{C}, \tag{4.16}$$

where $\varepsilon_n\to 0$ as $n\to\infty$. Thus

$$U([\mu]_b, z)\leqslant \liminf_{\substack{n\to\infty\\ n\in\Lambda_1}} U(\nu(\hat{p}_n), z), \qquad z\in\mathbf{C}. \tag{4.17}$$

Since the measures $\nu(\hat{p}_n)$ are all supported on a fixed compact set $\bar{\mathbf{C}}\backslash D_\infty(\mathscr{S}_1)$, by the strong version of the principle of descent (cf. [10, Theorem 3.8]), the right-hand side of (4.17) is q.e. equal to $U(\nu^*, z)$. Thus

$$U([\mu]_b, z)\leqslant U(\nu^*, z) \qquad \text{q.e. on } \mathbf{C}. \tag{4.18}$$

Since $U([\mu]_b, z)-U(\nu^*, z)$ is harmonic in $D_\infty(\mathscr{S})$ and vanishes at ∞, the maximum principle yields

$$U([\mu]_b, z) = U(\nu^*, z), \qquad z\in D_\infty(\mathscr{S}). \tag{4.19}$$

Note that since $U(\mu, z)=U([\mu]_b, z)$ for $z\in D_\infty(\mathscr{S})$, we have $U(\mu, z)=U(\nu^*, z)$ for $z\in D_\infty(\mathscr{S})$. Thus from Proposition 4.3 and the lower semicontinuity of $U(\nu^*, z)$, we obtain

$$U(\nu^*, z)\leqslant U(\mu, z) \qquad \text{q.e. on } \partial_\infty\mathscr{S}. \tag{4.20}$$

Therefore, from the definition of balayage,

$$U(\nu^*, z) \leqslant U([\mu]_b, z) \qquad \text{q.e. on } \partial_\infty \mathscr{S}. \tag{4.21}$$

From (4.19), (4.18), and (4.21) we get

$$U([\mu]_b, z) = U(\nu^*, z) \qquad \text{q.e. on } \overline{D_\infty(\mathscr{S})},$$

and so $[\mu]_b$ is a balayage (of finite energy) of ν^* to $\partial_\infty \mathscr{S}$. Consequently, $[\mu]_b = [\nu^*]_b$ by Theorem 2.2(ii).

We remark that *any* balayage of ν^* to $\partial_\infty \mathscr{S}$ has finite energy (and so $[\mu]_b$ is its unique balayage). Indeed, it is known (cf. [21]) that there is a (finite) constant M such that $U(\mu, z) \leqslant M$ for all $z \in \mathbf{C}$. If $\hat{\nu}^*$ is a balayage of ν^*, then

$$U(\hat{\nu}^*, z) = U(\nu^*, z) = U(\mu, z) \leqslant M, \qquad z \in D_\infty(\mathscr{S}),$$

and so, by the lower semi-continuity of $U(\hat{\nu}^*, z)$,

$$U(\hat{\nu}^*, z) \leqslant M, \qquad z \in \partial_\infty \mathscr{S},$$

which implies that $\hat{\nu}^*$ has finite energy. ∎

Proof of Theorem 2.3(b). Let ν^* be any weak-star limit measure of $\{\nu(p_n)\}_{n \in \Lambda}$. In view of (2.15) and (2.18), ν^* is supported on $\partial_\infty \mathscr{S}$. Thus by the above remark, ν^* has finite energy and so, by the maximum principle (cf. [10]), inequality (4.20) holds for all $z \in \mathbf{C}$. Next, using (2.17) and an argument similar to the one leading to (4.18), we deduce that

$$U(\mu, z) \leqslant U(\nu^*, z), \qquad \text{q.e. on } \mathbf{C}.$$

Thus, $U(\mu, z) = U(\nu^*, z)$ q.e. on $\mathbf{C}$, which implies that $\mu = \nu^*$. Since ν^* is an arbitrary limit measure, (2.19) follows. ∎

Proof of Theorem 2.3(c). Choose $\Lambda_1 \subset \Lambda$ such that

$$\limsup_{\substack{n \in \Lambda \\ n \to \infty}} \|w^n p_n\|_{\partial_\infty \mathscr{S}}^{1/n} = \lim_{\substack{n \to \infty \\ n \in \Lambda_1}} \|w^n p_n\|_{\partial_\infty \mathscr{S}}^{1/n}, \tag{4.22}$$

and let $z_n \in \partial_\infty \mathscr{S}$ satisfy

$$|w(z_n)^n p_n(z_n)| = \|w^n p_n\|_{\partial_\infty \mathscr{S}}. \tag{4.23}$$

Since $\partial_\infty \mathscr{S}$ is compact, there exist a subsequence $\Lambda_2 \subset \Lambda_1$ and a point $z_0 \in \partial_\infty \mathscr{S}$ such that $z_n \to z_0$ as $n \to \infty$, $n \in \Lambda_2$. Furthermore, let ν^* be a weak-star limit measure of $\{\nu(p_n)\}_{n \in \Lambda_2}$, say $\nu(p_n) \to \nu^*$ as $n \to \infty$, $n \in \Lambda_3 \subset \Lambda_2$. We note that since ν^* has a balayage satisfying (2.20), then as in the remark at the end of the proof of Theorem 2.3(a), it follows that ν^* has a unique balayage to $\partial_\infty \mathscr{S}$(namely, $[\mu]_b$). Thus, since $\partial_\infty \mathscr{S}$ is regular, we have

$$U(\nu^*, z) = U([\nu^*]_b, z) \qquad \text{for all} \quad z \in \overline{D_\infty(\mathscr{S})}. \tag{4.24}$$

Of course, (4.24) also holds with ν^* replaced by μ.

We now use, in order, the equations (4.22), (4.23), the principle of descent, (4.24), (2.20), and Proposition 4.3 to conclude that

$$\begin{aligned}
&\limsup_{\substack{n \to \infty \\ n \in \Lambda}} \frac{1}{n} \log \|w^n p_n\|_{\partial_\infty \mathscr{S}} \\
&\quad = \lim_{\substack{n \to \infty \\ n \in \Lambda_3}} \left[\int \log |z_n - t| \, d\nu(p_n)(t) - Q(z_n) \right] \\
&\quad \leqslant \int \log |z_0 - t| \, d\nu^*(t) - Q(z_0) \\
&\quad = -U([\nu^*]_b, z_0) - Q(z_0) \\
&\quad = -U([\mu]_b, z_0) - Q(z_0) \\
&\quad = -U(\mu, z_0) - Q(z_0) = -F.
\end{aligned} \tag{4.25}$$

Next, we use Proposition 4.2 (with p_n instead of P and $\|w^n p_n\|_{\partial_\infty \mathscr{S}}$ for M) and let $z \to \infty$ in (4.7) to conclude that

$$\exp(-F) \leqslant \liminf_{\substack{n \to \infty \\ n \in \Lambda}} \|w^n p_n\|_{\partial_\infty \mathscr{S}}^{1/n}.$$

Together with (4.25), we obtain that (2.14) holds with equality. ∎

The proof of Theorem 2.4 is similar to that of Theorem 2.3(a), except that z_0 now plays the role of the point at infinity. First we establish

LEMMA 4.4. *With the assumptions of Theorem 2.4, let $\kappa(n)$ be the precise degree of p_n. Then*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \kappa(n)/n = 1. \tag{4.26}$$

Moreover, for any closed set $A \subset D_\infty(\mathscr{S})$

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(p_n)(A) = 0. \tag{4.27}$$

Proof. We proceed as in the proof of Lemma 4.1. Let $G := D_\infty(\mathscr{S})$, $A \subset G$ be closed, and set

$$t_n(z) := \log|p_n(z)| + nU([\mu]_b, z) + (n - \kappa(n))\, g(z; \infty, G) + \sum_{j \in I} g(z; z_j, G), \tag{4.28}$$

where $\{z_j\}_1^{\kappa(n)}$ are the zeros of p_n and $I := \{j : 1 \leqslant j \leqslant \kappa(n), z_j \in A\}$. Then the function $t_n(z)$ is subharmonic in G (including ∞). From (2.21) and Proposition 4.2 we get that

$$\limsup_{\substack{z \to \zeta \\ z \in G}} t_n(z) \leqslant n(F + \varepsilon_n), \qquad \text{for q.e.} \quad \zeta \in \partial_\infty \mathscr{S},$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Thus, by the maximum principle,

$$t_n(z_0) \leqslant n(F + \varepsilon_n). \tag{4.29}$$

Let

$$m = m(A) := \min\{g(z_0; \infty, G), \quad \inf_{z \in A} g(z; z_0, G)\}.$$

Then $m > 0$ and, from (4.28), we see that

$$t_n(z_0) \geqslant \log|p_n(z_0)| + nU(\mu, z_0) + m(n - \kappa(n)) + m\kappa(n)\, \nu(p_n)(A),$$

and so, from (4.29), we obtain

$$\frac{1}{n} \log|p_n(z_0)| + U(\mu, z_0) + m\left(1 - \frac{\kappa(n)}{n}\right) + m\frac{\kappa(n)}{n} \nu(p_n)(A) \leqslant F + \varepsilon_n.$$

Applying hypothesis (2.22), it follows that

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \left[\left(1 - \frac{\kappa(n)}{n}\right) + \frac{\kappa(n)}{n} \nu(p_n)(A)\right] = 0,$$

which yields (4.26) and (4.27). ∎

Proof of Theorem 2.4. Let ν^* be a weak-star limit measure of $\{\nu(p_n)\}_{n \in \Lambda}$, say $\nu(p_n) \to \nu^*$ as $n \to \infty$, $n \in \Lambda_1 \subset \Lambda$. By Lemma 4.4, we have $\nu^* \in \mathscr{M}(\mathrm{Pc}(\mathscr{S}))$. Let $\overline{\Delta(z_0)} \subset D_\infty(\mathscr{S})$ be a closed neighborhood of z_0 having radius less than one. Applying Lemma 4.4 it is easy to see that there exist polynomials $\hat{p}_n \in \Pi_n$ such that

(i) $\deg \hat{p}_n = \deg p_n =: \kappa(n)$, $n \in \Lambda_1$;

(ii) $\hat{p}_n(z) \neq 0$ for $z \in \overline{\Delta(z_0)}$, $n \in \Lambda_1$;

(iii) $\{\hat{p}_n\}_{n \in \Lambda_1}$ satisfies (2.21) and (2.22);

(iv) $\nu(\hat{p}_n) \to \nu^*$ as $n \to \infty$, $n \in \Lambda_1$.

From (2.21) and Proposition 4.2 (with $P = \hat{p}_n$) we obtain

$$U([\mu]_b, z) - F \leqslant \varepsilon_n - \frac{1}{n} \log|\hat{p}_n(z)|, \qquad z \in \mathbf{C}, n \in \Lambda_1, \tag{4.30}$$

where $\varepsilon_n \to 0$ as $n \to \infty$. Since (4.30) can be written in the equivalent form

$$U(\mu, z_0) + \int \log \left|\frac{z-t}{z_0 - t}\right|^{-1} d[\mu]_b(t) - F + \frac{1}{n} \log|\hat{p}_n(z_0)|$$
$$\leqslant \varepsilon_n + \frac{\kappa(n)}{n} \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu(\hat{p}_n)(t),$$

we obtain from (2.22) and Lemma 4.4 that

$$\int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d[\mu]_b(t)$$
$$\leqslant \liminf_{\substack{n \to \infty \\ n \in \Lambda_1}} \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu(\hat{p}_n)(t), \qquad z \in \mathbf{C}. \tag{4.31}$$

Next, we claim that for quasi-every $z \in \mathbf{C}$

$$\liminf_{\substack{n \to \infty \\ n \in \Lambda_1}} \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu(\hat{p}_n)(t) = \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu^*(t). \tag{4.32}$$

This follows from the strong version of the principle of descent (cf. [10, Theorem 3.8]) as we now show. Let

$$q_n(\zeta) := \zeta^{\kappa(n)} \hat{p}_n\left(\frac{1}{\zeta} + z_0\right) \in \Pi_{\kappa(n)}.$$

Then

$$U\left(\nu(q_n), \frac{1}{z-z_0}\right) = \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu(\hat{p}_n)(t) + \log|z - z_0|,$$

and since $\nu(\hat{p}_n) \to \nu^*$, it is easy to verify that $\nu(q_n) \to \nu^{**}$, where $d\nu^{**}(s) = d\nu^*(s^{-1} + z_0)$. Since the $\hat{p}_n$'s do not vanish in a neighborhood of z_0, the measures $\nu(q_n)$ are all supported on a fixed compact subset of $\mathbf{C}$.

Hence by the principle of descent, we have for quasi-every $z \in \mathbf{C}$

$$\begin{aligned}\liminf_{n \to \infty} \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu(\hat{p}_n)(t) &= \liminf_{n \to \infty} U\left(\nu(q_n), \frac{1}{z-z_0}\right) - \log|z-z_0| \\ &= U\left(\nu^{**}, \frac{1}{z-z_0}\right) - \log|z-z_0| \\ &= \int \log \left|\frac{z-t}{z_0-t}\right|^{-1} d\nu^*(t),\end{aligned}$$

which establishes (4.32).

From (4.31) and (4.32) we have

$$U([\mu]_b, z) - U([\mu]_b, z_0) \leqslant U(\nu^*, z) - U(\nu^*, z_0) \qquad \text{q.e. on } \mathbf{C}.$$

Thus, since $U([\mu]_b, z) - U(\nu^*, z)$ is harmonic in $D_\infty(\mathcal{S})$, the maximum principle yields

$$U([\mu]_b, z) - U(\nu^*, z) = U([\mu]_b, z_0) - U(\nu^*, z_0), \qquad z \in D_\infty(\mathcal{S}).$$

But $U([\mu]_b, z) - U(\nu^*, z)$ vanishes at ∞, and so

$$U([\mu]_b, z) \leqslant U(\nu^*, z) \qquad \text{q.e. on } \mathbf{C}$$

and

$$U([\mu]_b, z) = U(\nu^*, z), \qquad z \in D_\infty(\mathcal{S}).$$

The remainder of the proof is identical to that of Theorem 2.3(a). ∎

Proof of Theorem 3.2. This is an immediate consequence of Theorem 2.3(a) with $w \equiv 1$. Indeed, for the equilibrium measure μ on E we have $\operatorname{cap}(\mathcal{S}) = \operatorname{cap}(E)$ and $\exp(-F) = \operatorname{cap}(E)$; thus (3.8) yields (2.14). ∎

Proof of Theorem 3.3. In order to prove Theorem 3.3, we recall (cf. [15, 16]) that for the weight $w_\alpha(x) := \exp(-|x|^\alpha/\lambda_\alpha)$, we have

$$\mathcal{S} = [-1, 1]; \qquad F = \log 2 + \frac{1}{\alpha}. \tag{4.33}$$

For p_n^* defined by (3.14), write $p_n^*(x) = a_n x^n + \cdots$. When $a_n \neq 0$, we set

$$\tilde{p}_n := a_n^{-1} p_n^*, \qquad \tilde{q}_n := \left(\frac{n}{\lambda_\alpha}\right)^{-n/\alpha} a_n^{-1} q_n^*, \tag{4.34}$$

where q_n^* is defined by (3.15). Then $\tilde{p}_n$ and $\tilde{q}_n$ are monic. Thus, in view of Theorem 2.3, it is enough to show that there exists a subsequence Λ of positive integers such that $a_n \neq 0$ if $n \in \Lambda$ and

$$\limsup_{\substack{n \to \infty \\ n \in \Lambda}} \|w_\alpha^n \tilde{q}_n\|_{\mathbf{R}}^{1/n} = (2e^{1/\alpha})^{-1}. \tag{4.35}$$

Now it is easy to see that

$$\limsup_{n \to \infty} \|W_\alpha p_n^*\|_{\mathbf{R}}^{1/n} = \limsup_{n \to \infty} \|w_\alpha^n q_n^*\|_{\mathbf{R}}^{1/n} \leqslant 1.$$

Hence (4.35) will be proved if we can show that

$$\limsup_{n \to \infty} \left[|a_n|^{1/n} \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha} \right] = 2e^{1/\alpha}. \tag{4.36}$$

For this purpose, we set

$$\mathscr{E}_{n,\alpha} := \min_{P \in \Pi_{n-1}} \|W_\alpha(x)(x^n - P(x))\|_{\mathbf{R}} =: \|W_\alpha T_n\|_{\mathbf{R}},$$

where $T_n(x) = x^n + \cdots \in \Pi_n$. It is known (cf. [15]) that

$$\left(\frac{n}{\lambda_\alpha}\right)^{-n/\alpha} \mathscr{E}_{n,\alpha} \geqslant \left(\frac{1}{2} e^{-1/\alpha}\right)^n, \qquad n = 1, 2, ..., \tag{4.37}$$

and

$$\lim_{n \to \infty} \left(\frac{n}{\lambda_\alpha}\right)^{-1/\alpha} \mathscr{E}_{n,\alpha}^{1/n} = \frac{1}{2} e^{-1/\alpha}. \tag{4.38}$$

Next, we observe that from (4.38)

$$\begin{aligned}
&\limsup_{n \to \infty} \left\{ |a_n|^{1/n} \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha} \right\} \\
&\quad \leqslant \limsup_{n \to \infty} \left\{ |a_n|^{1/n} \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha} \cdot \left(\frac{n}{\lambda_\alpha}\right)^{-1/\alpha} \mathscr{E}_{n,\alpha}^{1/n} \right\} \cdot 2e^{1/\alpha} \\
&\quad \leqslant \limsup_{n \to \infty} \left\{ |a_n|^{1/n} \|W_\alpha \tilde{p}_n\|_{\mathbf{R}}^{1/n} \right\} \cdot 2e^{1/\alpha} \\
&\quad \leqslant \limsup_{n \to \infty} \|W_\alpha p_n^*\|_{\mathbf{R}}^{1/n} \cdot 2e^{1/\alpha} \\
&\quad \leqslant 2e^{1/\alpha}.
\end{aligned} \tag{4.39}$$

Since $\alpha \geqslant 1$, expressions of the form $W_\alpha P$, where P is a polynomial, are dense in $C_0(\mathbf{R})$ (cf. [24]). So,

$$\lim_{n\to\infty} \varepsilon_n(\alpha, f) = 0.$$

It is then elementary to verify that (3.18) implies that

$$\limsup_{n\to\infty} [\varepsilon_{n-1}(\alpha, f) - \varepsilon_n(\alpha, f)]^{1/n} \geqslant 1.$$

Now, if we put

$$R_{n-1} := p_n^* - a_n T_n \in \Pi_{n-1},$$

then

$$\begin{aligned} 1 &\leqslant \limsup_{n\to\infty} [\varepsilon_{n-1}(\alpha, f) - \varepsilon_n(\alpha, f)]^{1/n} \\ &\leqslant \limsup_{n\to\infty} [\| W_\alpha(f - R_{n-1})\|_{\mathbf{R}} - \| W_\alpha(f - p_n^*)\|_{\mathbf{R}}]^{1/n} \\ &\leqslant \limsup_{n\to\infty} \| W_\alpha(p_n^* - R_{n-1})\|_{\mathbf{R}}^{1/n} \\ &= \limsup_{n\to\infty} \left\{ |a_n|^{1/n} \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha} \cdot \left(\frac{n}{\lambda_\alpha}\right)^{-1/\alpha} \| W_\alpha T_n\|_{\mathbf{R}}^{1/n} \right\} \\ &= (2e^{1/\alpha})^{-1} \cdot \limsup_{n\to\infty} \left\{ |a_n|^{1/n} \left(\frac{n}{\lambda_\alpha}\right)^{1/\alpha} \right\}. \end{aligned} \tag{4.40}$$

The estimates (4.40) and (4.39) prove (4.36). In turn, we have observed earlier that (4.36) implies the convergence of $\nu(q_n^*)$ to $\mu(w_\alpha, \mathbf{R})$.

The fact that (3.18) is satisfied if f is not an entire function, follows from Corollary 4 in [12]. In [12], this is proved only for a class of weight functions which includes W_α only when $\alpha \geqslant 2$. However, later results due to Rakhmanov [20] show (cf. Theorem 2, Corollary 3 in [12]) that Corollary 4 of [12] is also true if $\alpha > 1$. ▮

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